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On wQN_* and wQN^* spaces ☆

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ABSTRACT

Several properties of a topological space related to the behavior of open coverings or to the behavior of sequences of continuous real-valued functions defined on the space were recently studied. We modify some notions replacing continuous functions by lower or upper semicontinuous ones and we show that some of so obtained notions coincide with other (continuous case) notions.

1. Introduction

Recently several properties of a topological space related to the convergence of sequences of real-valued functions and some covering properties have been investigated. It turned out that some of them, originally introduced for essentially different purposes, are equivalent. We recall some of most important results and then we shall study some modifications of the considered notions.

Let X be a perfectly normal topological space with a countable base. Readers who feel more comfortable with separable metric spaces may assume that all spaces have these properties. We can tacitly assume that X is infinite since almost all considered properties are trivial for finite spaces. We deal with real valued functions from X into the closed unit interval $\langle 0, 1\rangle$. "$f_n \to 0$ on X" means that the sequence $\{f_n\}_{n=0}^{\infty}$ converges on X to 0 pointwise, i.e. for every $x \in X$ the sequence of reals $\{f_n(x)\}_{n=0}^{\infty}$ converges to 0. A sequence of such functions $\{f_n\}_{n=0}^{\infty}$ *quasi-normally converges* to f on a set X, written $f_n \xrightarrow{\mathrm{QN}} f$ on X, if there exists a sequence of positive reals $\varepsilon_n \to 0$ (a *control*) such that

$$(\forall x \in X)\ (\exists k)\ (\forall n > k)\quad \left|f_n(x) - f(x)\right| \leqslant \varepsilon_n.$$

A topological space X is a QN-*space* if every sequence of continuous functions converging to 0 on X converges quasi-normally on X. A topological space X is a wQN-*space* if from every sequence of continuous functions converging to 0 on X one can choose a subsequence converging to 0 quasi-normally on X—see [3]. X has the *sequence selection property*,

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shortly *SSP*, if for any sequence of sequences $\{f_{n,m}\}_{m=0}^{\infty}$, $n \in \omega$, of continuous functions such that $f_{n,m} \to 0$ on X for every n, there exists a sequence $\{m_n\}_{n=0}^{\infty}$ such that $f_{n,m_n} \to 0$ on X. Actually one can ask that the sequence $\{m_n\}_{n=0}^{\infty}$ is unbounded or even increasing. The basic result is

Theorem 1 *(M. Scheepers–D. Fremlin). A topological space X has the sequence selection property if and only if X is a* wQN*-space.*

M. Scheepers [11] proved the implication from left to right and D. Fremlin [5] proved the opposite implication.

We assume that the reader is familiar with the properties (α_1)–(α_4) introduced by A.V. Arkhangel'skiĭ [1]. $\mathrm{C}_p(X)$ denotes the set of all continuous real-valued functions defined on X with the topology inherited from the product space ${}^X\mathbb{R}$. A sequence f_n converges to f in this topology if and only if f_n converges to f pointwise on X. One can easily show that a topological space X has the sequence selection property if and only if $\mathrm{C}_p(X)$ has property (α_2). Thus:

Corollary 2. *X is a* wQN*-space if and only if $\mathrm{C}_p(X)$ has property (α_2).*

Moreover:

Theorem 3. *(See M. Scheepers [10].)*

(1) *$\mathrm{C}_p(X)$ has property (α_2) if and only if $\mathrm{C}_p(X)$ has property (α_4).*
(2) *If $\mathrm{C}_p(X)$ has property (α_1) then X is a* QN*-space.*

Theorem 4. *(See M. Sakai [9] and L. Bukovský and J. Haleš [2].) If X is a* QN*-space then $\mathrm{C}_p(X)$ has property (α_1).*

Corollary 5. *X is a* QN*-space if and only if $\mathrm{C}_p(X)$ has property (α_1).*

A family $\mathcal{U} \subseteq \mathcal{P}(X)$ is a *cover of* X if $X = \bigcup \mathcal{U}$ and $X \notin \mathcal{U}$. An infinite cover $\mathcal{U}$ is a γ*-cover* if every $x \in X$ lies in all but finitely many members of $\mathcal{U}$. A γ-cover $\mathcal{U}$ is *shrinkable*, if there exists a closed γ-cover $\mathcal{V}$ that is a refinement of $\mathcal{U}$. $\mathrm{O}(X)$, $\Gamma(X)$, $\Gamma^{\mathrm{co}}(X)$, and $\Gamma^{\mathrm{sh}}(X)$ will denote the set of all countable open covers, countable open γ-covers, countable clopen γ-covers, and countable open shrinkable γ-covers of X, respectively. Let $\mathcal{A}$, $\mathcal{B}$ be families of covers of a topological space X. The space X is a $\mathrm{S}_1(\mathcal{A},\mathcal{B})$*-space* if for every sequence $\{\mathcal{U}_n\}_{n=0}^{\infty}$ of covers from $\mathcal{A}$ there exist sets $U_n \in \mathcal{U}_n$ such that $\{U_n;\ n \in \omega\} \in \mathcal{B}$.

Theorem 6. *(See M. Scheepers [11].) $\mathrm{S}_1(\Gamma,\Gamma)$-space is a* wQN*-space.*

Conjecture 1 *(M. Scheepers). Perfectly normal* wQN*-space has property $\mathrm{S}_1(\Gamma,\Gamma)$.*

By results of [2] and [9]

Theorem 7. *Every perfectly normal* QN*-space is a $\mathrm{S}_1(\Gamma,\Gamma)$-space.*

B. Tsaban and L. Zdomskyy [12] have realized that according to the result of A. Dow [4] saying that (α_2) is equivalent to (α_1) in Laver's model, we obtain

Theorem 8. *In Laver's model* [8]*, a topological space X is a* wQN*-space if and only if X is a* QN*-space. Consequently, Scheepers conjecture is consistent with* ZFC*.*

Let us note that in [6] the authors essentially show

Theorem 9. *If $\mathfrak{t} = \mathfrak{b}$, then there exists a $\mathrm{S}_1(\Gamma,\Gamma)$-subset of the Baire space ${}^{\omega}\omega$ which is not a* QN*-space. Consequently, there exists a* wQN *which is not a* QN*-space.*

However the following problem is still open.

Problem 1. Is the inequality $\mathrm{S}_1(\Gamma,\Gamma) \neq \mathrm{wQN}$ consistent with *ZFC*?

The best known result toward the problem is the following:

Theorem 10. *For a perfectly normal topological space X, the following are equivalent*:

(a) *X is a* wQN*-space,*
(b) *X is a* $\mathrm{S}_1(\Gamma^{\mathrm{sh}}, \Gamma)$*-space,*
(c) $\mathrm{Ind}(X) = 0$ *and X is a* $\mathrm{S}_1(\Gamma^{\mathrm{co}}, \Gamma)$*-space.*

The equivalence of (a) and (b) is proved in [2], the equivalence of (a) and (c) is proved in [9] and essentially in [7].

Now we shall present some modifications of the introduced notions and corresponding results. Let us recall that a function $f : X \to \mathbb{R}$ is *lower* (*upper*) *semicontinuous* if for every real a the set $\{x \in X\colon\ f(x) > a\}$ (the set $\{x \in X\colon\ f(x) < a\}$) is open. If in definitions of a wQN-space and SSP we replace continuous functions by lower or upper semicontinuous ones, we obtain definitions of the notions which we shall call QN_*-space, wQN_*-space, SSP_* or QN^*-space, wQN^*-space, SSP^*, respectively. E.g. X is a wQN_**-space* if from every sequence of lower semicontinuous functions converging to 0 on X one can choose a subsequence converging quasi-normally to 0 on X.

Theorem 11. $\mathrm{wQN}_* \equiv \mathrm{SSP}_*$.

Proof. Let $f_{n,m}$, $n, m \in \omega$ be lower semicontinuous, $f_{n,m} \to 0$ on X for every n. Set

$$g_m(x) = \sum_{n=0}^{\infty} \min\bigl\{2^{-n}, f_{n,m}(x)\bigr\}.$$

Since

$$\bigl\{x \in X\colon\ g_m(x) > a\bigr\} = \bigcup_k \left\{x \in X\colon\ \sum_{n=0}^{k} \min\bigl\{2^{-n}, f_{n,m}(x)\bigr\} > a\right\},$$

one can show that the function g_m is lower semicontinuous and $g_m \to 0$ on X. Therefore there exists an increasing sequence $\{m_n\}_{n=0}^{\infty}$ of natural numbers such that $g_{m_n} \xrightarrow{\mathrm{QN}} 0$ on X with the control $\{2^{-n}\}_{n=0}^{\infty}$. If $g_{m_n}(x) < 2^{-n}$ then

$$\min\bigl\{2^{-n}, f_{n,m_n}(x)\bigr\} = f_{n,m_n}(x) < 2^{-n}.$$

Hence $f_{n,m_n} \to 0$ on X.

If $f_m \to 0$ on X are lower semicontinuous then $2^n f_m(x) \to 0$ for every n. By SSP_* there exists an increasing sequence $\{m_n\}_{n=0}^{\infty}$ such

$$2^n f_{m_n} \to 0 \quad \text{on } X.$$

Then $f_{m_n} \xrightarrow{\mathrm{QN}} 0$ on X with control $\{2^{-n}\}_{n=0}^{\infty}$. □

Theorem 12. $\mathrm{SSP}^* \to \mathrm{wQN}^*$.

Proof is the same as the proof of $\mathrm{SSP}_* \to \mathrm{wQN}_*$.

Problem 2. $\mathrm{wQN}^* \to \mathrm{SSP}^*$?

Theorem 13. $\mathrm{S}_1(\Gamma, \Gamma) \equiv \mathrm{SSP}^*$.

Proof. Assume that $f_{n,m}$ are upper semicontinuous and $f_{n,m} \to 0$ on X for every fixed n. We set

$$U_{n,m} = \bigl\{x \in X\colon\ 2^n f_{n,m}(x) < 1\bigr\}.$$

Then every $U_{n,m}$ is open and we can assume that no $U_{n,m}$ is equal to the whole space X. Then $\{U_{n,m}\colon\ m \in \omega\}$ is a γ-cover for every fixed n. By $\mathrm{S}_1(\Gamma, \Gamma)$ there exists a sequence $\{m_n\}_{n=0}^{\infty}$ such that $\{U_{n,m_n}\colon\ n \in \omega\}$ is a γ-cover. Moreover, we can assume that $\{m_n\}_{n=0}^{\infty}$ is unbounded. Then $f_{n,m_n} \to 0$ on X.

Assume now that $\{U_{n,m}\colon\ m \in \omega\}$ is a γ-cover of X for every $n \in \omega$. We set

$$f_{n,m}(x) = \begin{cases} 0 & \text{if } x \in U_{n,m}, \\ 1 & \text{otherwise.} \end{cases}$$

Then every $f_{n,m}$ is upper semicontinuous and $f_{n,m} \to 0$ on X for every fixed n. By SSP^* there exists an increasing sequence $\{m_n\}_{n=0}^{\infty}$ such that $f_{n,m_n} \to 0$ on X. One can easily see that $\{U_{n,m_n}\colon\ n \in \omega\}$ is a γ-cover of X. □

Theorem 14. $\mathrm{wQN}_* \to \mathrm{QN}$.

Proof. Assume that f_n, $n \in \omega$ are continuous functions defined on X. We set

$$g_n(x) = \sup\{f_m(x):\ m \geqslant n\}.$$

Then g_n is a lower semicontinuous function on X for every n. Moreover, if $f_n \to 0$ on X then also $g_n \to 0$ on X. Since X is a wQN$_*$-space, there exists an increasing sequence $\{n_m\}_{m=0}^{\infty}$ such that $g_{n_m} \xrightarrow{\text{QN}} 0$ on X with control 2^{-m}. Note that $f_m(x) \leqslant g_n(x)$ for $m \geqslant n$. If we set $\varepsilon_k = 2^{-m}$ for $n_{m-1} < k \leqslant n_m$, then $f_n \xrightarrow{\text{QN}} 0$ on X with control ε_n. $\square$

B. Tsaban and L. Zdomskyy [12] have proved that X is a QN-space if and only if each sequence of Borel measurable functions converging to 0 on X converges to 0 also quasi-normally. Since a lower or upper semicontinuous function is Borel measurable we obtain

Corollary 15. $\text{QN} = \text{wQN}_*$, $\text{QN} = \text{QN}^* = \text{QN}_*$.

By Theorems 7, 11–14 we obtain

Corollary 16. $\text{SSP}_* \to \text{SSP}^*$, $\text{wQN}_* \to \text{wQN}^*$.

By Theorems 8 and 9 the opposite implications are undecidable in *ZFC*.
We summarize our results in the following picture.

$$\begin{tikzcd}
\text{QN} \arrow[r] \arrow[dd, leftrightarrow] & \text{S}_1(\Gamma,\Gamma) \arrow[d] \arrow[ddr, leftrightarrow] & \\
 & \text{S}_1(\Gamma^{\text{sh}},\Gamma) \arrow[d, leftrightarrow] & \\
\text{SSP}_* \arrow[r] \arrow[d, leftrightarrow] & \text{SSP} \arrow[d, leftrightarrow] & \text{SSP}^* \arrow[l] \arrow[d] \\
\text{wQN}_* \arrow[r] & \text{wQN} & \text{wQN}^* \arrow[l]
\end{tikzcd}$$

So we can restrict our considerations to the following four notions:

$$\text{QN} \to \text{S}_1(\Gamma,\Gamma) \to \text{wQN}^* \to \text{wQN}.$$

By Theorem 8 in Laver's model all those four notions are equal. By Theorem 9 *ZFC* is consistent with $\text{QN} \neq \text{S}_1(\Gamma,\Gamma)$ and therefore with $\text{QN} \neq \text{wQN}$.

Problem 3. Is any of the inequalities

$$\text{S}_1(\Gamma,\Gamma) \neq \text{wQN}^*, \qquad \text{S}_1(\Gamma,\Gamma) \neq \text{wQN}, \qquad \text{wQN}^* \neq \text{wQN}$$

consistent with *ZFC*?

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